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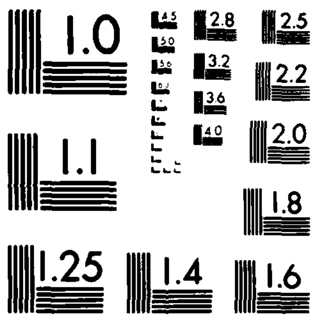


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AN INTEGRAL INEQUALITY WITH APPLICATIONS
TO ORDER STATISTICS

by

Philip J. Boland and Frank Proschan

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$$\int_x^\infty \bar{G}(t)dt \geq \int_x^\infty \bar{F}(t)dt \qquad \text{for all } x \geq 0$$

and

$$\int_0^\infty \bar{G}(t)dt = \int_0^\infty \bar{F}(t)dt < +\infty .$$

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$$\int_0^\infty \psi(t)\Phi(\overline{G}_1(t),\ldots,\overline{G}_n(t))dt \le \int_0^\infty \psi(t)\Phi(\overline{F}_1(t),\ldots,\overline{F}_n(t))dt.$$

Applications in reliability theory and order statistics are given.

# AN INTEGRAL INEQUALITY WITH APPLICATIONS TO ORDER STATISTICS

by

Philip J. Boland and Frank Proschan


## ABSTRACT

We say the life distribution function G majorizes the life distribution function F (written $G \overset{m}{>} F$) if

$$\int_x^\infty \overline{G}(t)dt \geq \int_x^\infty \overline{F}(t)dt \qquad \text{for all } x \geq 0$$

$$\text{and} \qquad \int_0^\infty \overline{G}(t)dt = \int_0^\infty \overline{F}(t)dt < +\infty .$$

An integral inequality is proved giving sufficient conditions on functions $\psi$ and $\Phi$ in order to ensure that whenever $G_i \overset{m}{>} F_i$ for $i=1,\ldots, n$, then

$$\int_0^\infty \psi(t)\Phi(\overline{G}_1(t),\ldots,\overline{G}_n(t))dt \leq \int_0^\infty \psi(t)\Phi(\overline{F}_1(t),\ldots,\overline{F}_n(t))dt.$$

Applications in reliability theory and order statistics are given.

1. Introduction.

For given life distribution functions F and G, the respective survival functions are $\overline{F} = 1-F$ and $\overline{G} = 1-G$. We define the partial ordering $\overset{m}{>}$ on the class of life distributions with finite means by $G \overset{m}{>} F$ (m for majorization) if

$$(1.1) \qquad \int_x^\infty \overline{G}(t)dt \geq \int_x^\infty \overline{F}(t)dt \qquad \text{for all } x \geq 0$$

and

$$(1.2) \qquad \mu_G = \int_0^\infty \overline{G}(t)dt = \int_0^\infty \overline{F}(t)dt = \mu_F < +\infty .$$

If X and Y are nonnegative random variables with respective distribution functions F and G, then Ross [11] says "Y is more variable than X" (written $Y \geq_v X$ or $G \geq_v F$) if (1.1) holds. Stoyan [14] equivalently defines Y to be "larger in mean residual life" than X (written $G \geqq_c F$ or in previous publications $G \overset{(2)}{\geq} F$) if (1.1) holds. Bessler and Veinott [3] use the terminology "Y is stochastically larger in mean than X." The notation of Stoyan (c for convex) is suggested by the following characterization:

$$G \geqq_c F$$

$$\Longleftrightarrow$$

$$\int_0^\infty \Psi(t)dG(t) \geq \int_0^\infty \Psi(t)dF(t)$$

holds for all increasing (that is nondecreasing) convex functions $\Psi$, provided the integrals exist.

For life distribution functions F and G, $G \overset{m}{>} F$ if and only if $G \geqq_c F$ (or $G \geq_v F$) and G and F have equal finite means ($\mu_F = \mu_G$). For distribution functions with finite means, the following useful characterization of $G \overset{m}{>} F$ (see for example Ross [11] or Stoyan [14]) is an immediate corollary of Theorem 2.1:

$$G \overset{m}{>} F$$

$$\Longleftrightarrow$$

$$\int_0^\infty \Psi(t)dG(t) \geq \int_0^\infty \Psi(t)dF(t)$$

holds for all convex functions $\Psi$, provided the integrals exist.

We note in particular that if $G \overset{m}{>} F$, then

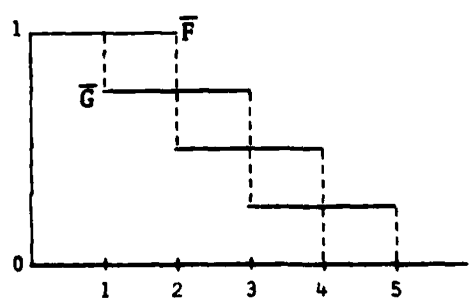
$$\sigma_G^2 = \int_0^\infty (t-\mu_G)^2 dG(t) \geq \int_0^\infty (t-\mu_F)^2 dF(t) = \sigma_F^2 \quad .$$

Hence $G \overset{m}{>} F$ implies that the life distribution represented by G is 'more dispersed' than that represented by F around their common mean.

For life distribution functions F and G with a common mean, $G \overset{m}{>} F$ is a more general relationship than $G \underset{*}{\geq} F$ (F is star shaped with respect to G). When F and G are continuous life distributions (where $F(0) = G(0) = 0$, F and G have interval support and G is strictly increasing on its support), then $G \underset{*}{\geq} F$ if $G^{-1}F(x)$ is star-shaped (that is $\frac{G^{-1}F(x)}{x}$ is increasing for $x > 0$). If $G \underset{*}{\geq} F$ and F and G have a common mean, then $\overline{F}(x)$ crosses $\overline{G}(x)$ once and from above as $x{:}0 \to \infty$, so that in particular $G \overset{m}{>} F$ (see Barlow and Proschan [2]). For a continuous life distribution function F with mean $\mu$, let us define $G(x) = 1-e^{-x/\mu}$ to be the exponential distribution with the same mean. Then F is IFRA (increasing failure rate average) $\Longleftrightarrow G \underset{*}{\geq} F$, and F is HNBUE (harmonic new better than used in expectation) $\Longleftrightarrow G \overset{m}{>} F$. See Klefsjö [6] for further properties of HNBUE distributions.

If F and G are two life distribution functions with common mean and $\overline{F}(x)$ crosses $\overline{G}(x)$ once and from above as $x{:}0 \to \infty$, then $G \overset{m}{>} F$, however the converse is clearly not true. For example let F and G be defined as follows:

$$F(x) = \begin{cases} 0 & x < 2 \\ 1/2 & 2 \leq x < 4 \\ 1 & 4 \leq x \end{cases} \qquad G(x) = \begin{cases} 0 & x < 1 \\ 1/4 & 1 \leq x < 3 \\ 3/4 & 3 \leq x < 5 \\ 1 & 5 \leq x \end{cases} \quad .$$




Then $G \overset{m}{>} F$ and G 'crosses' F three times.

A vector $\underline{b} = (b_1,\ldots,b_n)$ majorizes the vector $\underline{a} = (a_1,\ldots,a_n)$ if

$$\sum_{i=k}^{n} b_{[i]} \geq \sum_{i=k}^{n} a_{[i]} \qquad \text{for } k=2,\ldots,n$$

$$\text{and} \quad \sum_{i=1}^{n} b_{[i]} = \sum_{i=1}^{n} a_{[i]} ,$$

where the $b_{[i]}$'s and $a_{[i]}$'s are the components of $\underline{b}$ and $\underline{a}$ respectively in ascending order. When $\underline{b}$ majorizes $\underline{a}$ we write $\underline{b} \overset{m}{>} \underline{a}$.

Suppose now that $\underline{b}$ and $\underline{a}$ are n dimensional vectors with nonnegative components such that $\underline{b} \overset{m}{>} \underline{a}$. If G and F are respectively the distribution functions for the uniform distributions on the components of $\underline{b}$ and $\underline{a}$, then $G \overset{m}{>} F$. This is our motivation for using the letter m for our partial ordering on the family of life distribution functions with finite means.

## 2. An Integral Inequality.

The following theorem is a variant of an integral inequality obtained by Fan and Lorentz [4].

Theorem 2.1. Let $\Phi = [0,1]^n \to [0,\infty)$ be a continuous increasing function, and assume that for $i=1,\ldots,n$, $F_i$ and $G_i$ are life distribution functions where $G_i \overset{m}{>} F_i$.

a) If $\psi$ is nonnegative decreasing, $\Phi$ is convex in each variable separately and $\Phi$ satisfies the following property:

$$(2.1) \qquad \Phi(u_i+ h, u_j+ k) - \Phi(u_i+ h, u_j) - \Phi(u_i, u_j+ k) + \Phi(u_i, u_j) \geq 0$$

for all $i \neq j$, $0 \leq u_i \leq u_i+ h \leq 1$, $0 \leq u_j \leq u_j+ k \leq 1$
(where we have used the notational simplification of omitting those arguments of $\Phi$ which are the same in a given formula),
then providing the integrals exist,

$$(2.2) \qquad \int_0^\infty \psi(t)\Phi(\overline{G}_1(t),\ldots, \overline{G}_n(t))dt \leq \int_0^\infty \psi(t)\Phi(\overline{F}_1(t),\ldots, \overline{F}_n(t))dt.$$

b) If $\psi$ is nonnegative increasing, $\Phi$ is concave in each variable separately and $\Phi$ satisfies the following property:

$$\Phi(u_i+h, u_j+k) - \Phi(u_i+h, u_j) - \Phi(u_i, u_j+k) + \Phi(u_i, u_j) \le 0 \tag{2.3}$$

for all $i \ne j$, $0 \le u_i \le u_i + h \le 1$, $0 \le u_j \le u_j + k \le 1$,

then providing the integrals exist

$$\int_0^\infty \psi(t)\Phi(\overline{G}_1(t),\ldots,\overline{G}_n(t))dt \ge \int_0^\infty \psi(t)\Phi(\overline{F}_1(t),\ldots,\overline{F}_n(t))dt. \tag{2.4}$$

Proof: We prove only a), the proof of b) following in a similar fashion.

(i) Initially we show that it suffices to prove the result for the case when $F_1, G_1, \ldots, F_n, G_n$ all have finite support. In turn to establish this we show that if the inequality is valid whenever $F_1$ and $G_1$ have finite support, then it is true in general.

Suppose now that $F_1, G_1, \ldots, F_n, G_n$ are arbitrary life distributions where $G_i \overset{m}{>} F_i$ for $i = 1,\ldots,n$. Given $\varepsilon > 0$, we can find $S$ so that

$$\int_S^\infty \psi(t)\Phi(\overline{G}_1(t),\ldots,\overline{G}_n(t))dt < \varepsilon .$$

Now define $F_1'$ and $G_1'$ by

$$\overline{F}_1'(t) = \begin{cases} \overline{F}_1(t) & t < S \\ 0 & t \ge S \end{cases}$$

$$\overline{G}_1'(t) = \begin{cases} \overline{G}_1(t) & t < S \\ \overline{G}_1(S) & S \le t \le S + \dfrac{\int_0^S \overline{F}_1(t)dt - \int_0^S \overline{G}_1(t)dt}{\overline{G}_1(S)} \\ 0 & \text{otherwise,} \end{cases}$$

(if $\bar{G}_1(S) = 0$, then both $G_1$ and $F_1$ have finite support). Then $G_1' \overset{m}{>} F_1'$, and

$$\int_0^\infty \psi(t)\Phi(\bar{F}_1(t), \bar{F}_2(t),\ldots,\bar{F}_n(t))dt \geq \int_0^\infty \psi(t)\Phi(\bar{F}_1'(t), \bar{F}_2(t),\ldots,\bar{F}_n(t))dt$$

$$\geq \int_0^\infty \psi(t)\Phi(\bar{G}_1'(t), \bar{G}_2(t),\ldots,\bar{G}_n(t))dt$$

$$\geq \int_0^\infty \psi(t)\Phi(\bar{G}_1(t), \bar{G}_2(t),\ldots,\bar{G}_n(t))dt - \varepsilon .$$

Since $\varepsilon$ is arbitrary, the conclusion follows.

(ii) It now remains to show that

$$\int_0^\infty \psi(t)\Phi(\bar{G}_1(t),\ldots,\bar{G}_n(t))dt \leq \int_0^\infty \psi(t)\Phi(\bar{F}_1(t),\ldots,\bar{F}_n(t))dt$$

whenever $G_i \overset{m}{>} F_i$ for all $i = 1,\ldots, n$, and where the support of $F_i$ and $G_i \subset [0,S]$ for all $i = 1,\ldots, n$.

Let $\varepsilon > 0$ be given. As $\Phi$ is continuous, there exists a $\delta > 0$ such that whenever $u, v \in [0,1]^n$ and $\|u-v\| = \max_{i=1,\ldots,n} |u_i-v_i| < \delta$, then $|\Phi(u) - \Phi(v)| < \varepsilon/2S\psi(0)$.

There exist only a finite number of points r in [0,S] where at least one of $F_1$, $G_1,\ldots,F_n$, $G_n$ has a jump discontinuity with jump $> \delta/2$. Hence we can find an integer N large enough so that

(1) $\dfrac{\psi(0)4rS \sup|\Phi|}{\varepsilon} < N$

and (2) on all but at most r of the N intervals $\left[0,\frac{S}{N}\right)$,

$\left[\frac{S}{N}, \frac{S+1}{N}\right)$ $\ldots$, $\left[\frac{(N-1)S}{N}, \frac{NS}{N}\right)$,

$$\max_i \left[\bar{F}_i\left(\frac{jS}{N}\right) - \bar{F}_i\left(\frac{(j+1)S}{N}\right)\right] < \delta \text{ and } \max_i \left[\bar{G}_i\left(\frac{jS}{N}\right) - \bar{G}_i\left(\frac{(j+1)S}{N}\right)\right] < \delta .$$

Hence for each $i = 1,\dots,n$, we define the following simple survival functions:

$$\bar{F}_i''(t) = \left(\int_{jS/N}^{(j+1)S/N} \bar{F}_i(t)dt\right)\Big/ S/N$$

and

$$\bar{G}_i''(t) = \left(\int_{jS/N}^{(j+1)S/N} \bar{G}_i(t)dt\right)\Big/ S/N$$

when $t \in \left[\frac{jS}{N}, \frac{(j+1)S}{N}\right)$ for some $j = 0,\dots,N-1$, and zero otherwise.

Note that $G_i'' \overset{m}{>} F_i''$ for all $i = 1,\dots,n$.

Moreover,

$$\left| \int_0^S \psi(t)\Phi(\bar{F}_1(t),\dots,\bar{F}_n(t))dt - \int_0^S \psi(t)\Phi(\bar{F}_1''(t),\dots,\bar{F}_n''(t))dt \right|$$

$$= \left| \sum_{j=0}^{N-1} \int_{jS/N}^{(j+1)S/N} \psi(t)\left[\Phi(\bar{F}_1(t),\dots,\bar{F}_n(t))dt - \Phi(\bar{F}_1''(t),\dots,\bar{F}_n''(t))\right]dt \right|$$

$$< \psi(0)2r \sup|\Phi| \frac{S}{N} + \frac{\varepsilon}{2S}N\left(\frac{S}{N}\right)$$

$$< \varepsilon .$$

Similarly, $\left| \int_0^S \psi(t)\Phi(\bar{G}_1(t),\dots,\bar{G}_n(t))dt - \int_0^S \psi(t)\Phi(\bar{G}_1''(t),\dots,\bar{G}_n''(t))dt \right| < \varepsilon$ .

Therefore, it suffices to prove (2.2) for the case when all $F_i, G_i$ are step functions which are constant on $\left[\frac{jS}{N}, \frac{(j+1)S}{N}\right)$, $j = 0, \dots, N-1$. Furthermore, without loss of generality we may assume that $\psi$ is constant on each interval of the form $\left[\frac{jS}{N}, \frac{(j+1)S}{N}\right)$ for $j = 0, \dots, N-1$.

(iii) Assume now that $G_i \overset{m}{>} F_i$ for $i = 1,\ldots,n$ and that all 2n functions have support in $[0,S)$ and are constant on each interval $\left[\frac{jS}{N}, \frac{(j+1)S}{N}\right)$ for $j=0,\ldots,N-1$. We also assume $\psi$ is constant on each of these intervals and use the notational simplification $\psi(j) = \psi\left(\frac{jS}{N}\right)$ for $j=0,\ldots,N-1$.

Each $\overline{G}_i$ may be transformed into $\overline{F}_i$ by a finite succession of transformations $\tau$ of the following type (see Hardy, Littlewood and Pólya [5]). $\tau$ changes the value $v_{ji}$ of $\overline{G}_i$ on the interval $\left[\frac{jS}{N}, \frac{(j+1)S}{N}\right)$ into $v_{ji}+ h$ and the value $v_{ki}$ of $\overline{G}_i$ on $\left[\frac{kS}{N}, \frac{(k+1)S}{N}\right)$ into $v_{ki}- h$ where $j < k$ and

$$0 \le v_{ki}- h \le v_{ki} \le v_{ji} \le v_{ji} + h \le 1.$$

Letting $\Delta_\tau$ denote the change in the integral $\int_0^\infty \psi(t)\Phi(\overline{G}_1(t),\ldots,\overline{G}_n(t))dt$ resulting from such a transformation $\tau$, we complete the proof by showing that $\Delta_\tau \ge 0$. Without loss of generality $i=1$, and hence

$$\begin{aligned}
\Delta_\tau &= \frac{S}{N}\{\psi(j)[\Phi(v_{j1}+ h, v_{j2},\ldots,v_{jn}) - \Phi(v_{j1}, v_{j2},\ldots, v_{jn})] \\
&\quad - \psi(k)[(\Phi(v_{k1},v_{k2},\ldots,v_{kn}) - \Phi(v_{k1}- h,v_{k2},\ldots,v_{kn})]\} \\
&\ge \psi(k)\frac{S}{N}\{\Phi(v_{j1}+ h,v_{j2},\ldots,v_{jn}) - \Phi(v_{j1},v_{j2},\ldots,v_{jn}) \\
&\quad - (\Phi(v_{j1}+ h,v_{k2},\ldots,v_{kn}) - \Phi(v_{j1},v_{k2},\ldots,v_{kn}))\}
\end{aligned}$$

(since $\Phi$ is convex in each variable separately)

$$= \psi(k)\frac{S}{N}\{[\Phi(v_{j1}+h, v_{k2}+h_2,\ldots,v_{kn}+h_n) - \Phi(v_{j1},v_{k2}+h_2,\ldots v_{kn}+h_n)$$

$$- \Phi(v_{j1}+h, v_{k2}+h_2,\ldots,v_{kn}) + \Phi(v_{j1},v_{k2}+h_2,\ldots v_{kn})]$$

$$+\ldots$$

$$+ [\Phi(v_{j1}+h, v_{k2}+h_2, v_{k3},\ldots,v_{kn}) - \Phi(v_{j1},v_{k2}+h_2,v_{k3},\ldots,v_{kn})$$

$$- \Phi(v_{j1}+h, v_{k2},\ldots,v_{kn}) + \Phi(v_{j1},v_{k2},\ldots,v_{kn})]\}$$

$\geq 0$

(since $\Phi$ satisfies property (2.1) and $\psi$ is nonnegative).

Here $h_i = v_{ji} - v_{ki}$ for $i = 2,\ldots,n$.

Corollary 2.2. Let G and F be life distribution functions with finite means. Then $G \overset{m}{>} F$ if and only if

a) *For all nonnegative increasing* continuous convex $\Phi$ and nonnegative decreasing $\psi$,

$$\int_0^\infty \psi(t)\Phi(\overline{G}(t))dt \leq \int_0^\infty \psi(t)\Phi(\overline{F}(t))dt$$

and

b) For all nonnegative increasing continuous concave $\Phi$ and nonnegative increasing $\psi$,

$$\int_0^\infty \psi(t)\Phi(\overline{G}(t))dt \geq \int_0^\infty \psi(t)\Phi(\overline{F}(t))dt,$$

provided the integrals exist.

Proof. The only if part follows immediately from Theorem 2.1. Assume now a) and b) hold. Letting $\Phi(u)=u$ and $\psi_x(t) = \chi_{[x,+\infty)}$ (that is the characteristic function of the interval $[x,+\infty)$) it follows from b) that $\int_x^\infty \overline{G}(t)dt \geq \int_x^\infty \overline{F}(t)dt$ for all $x \geq 0$. Taking $\psi(t)\equiv 1$, it follows from a) that $\mu_F = \mu_G$.

Corollary 2.3. If G and F are life distributions with finite means, then

$G \overset{m}{>} F$

$\iff$

(2.5) $$\int_0^\infty \Psi(t)dG(t) \geq \int_0^\infty \Psi(t)dF(t)$$

holds for all convex functions $\Psi$, provided the integrals exist.

Proof. The if part of the result is immediate. Now suppose $G \overset{m}{>} F$. It suffices to prove (2.5) for the case where $\Psi$ has derivative $\psi$ and $\Psi(0)=0$.

Then
$$\begin{aligned}\int_0^\infty \Psi(t)dG(t) &= \int_0^\infty \psi(t)\overline{G}(t)dt \\ &= \int_0^\infty [\psi(t)-\psi(0)]\overline{G}(t)dt + \psi(0)\mu_G \\ &\geq \int_0^\infty [\psi(t)-\psi(0)]\overline{F}(t)dt + \psi(0)\mu_F \quad \text{(by Theorem 2.1)} \\ &= \int_0^\infty \Psi(t)dF(t).\end{aligned}$$

Remark 2.4. Another approach to (2.5) in the proof of Corollary 2.3 is as follows. Suppose $G \overset{m}{>} F$. Let $Z_G$ and $Z_F$ be the random variables with respective densities $\frac{1}{\mu_G}\int_0^x \overline{G}(t)dt$ and $\frac{1}{\mu_F}\int_0^x \overline{F}(t)dt$. Then $Z_G \overset{st}{\geq} Z_F$ ($Z_G$ is stochastically larger than $Z_F$) and hence (see for example Ross [11]) $E(\psi(Z_G)) \geq E(\psi(Z_F))$ for all increasing $\psi$. But

$$\int_0^\infty \psi(t)\overline{G}(t)dt = E(\psi(Z_G)) \geq E(\psi(Z_F)) = \int_0^\infty \psi(t)\overline{F}(t)dt.$$

## 3. Applications.

Theorem 3.1. Let $X_1,\ldots,X_n,Y_1,\ldots,Y_n$ be independent nonnegative random variables where $X_i \sim F_i$ and $Y_i \sim G_i$ for $i=1,\ldots,n$, and let $X_{[1]},\ldots, X_{[n]}$ and $Y_{[1]},\ldots,Y_{[n]}$ be respectively the X (Y) observations in increasing order.

Assume that $G_i \overset{m}{>} F_i$ for $i=1,\ldots,n$. Then

a) $\int_x^\infty P[Y_{[n]}+\ldots+Y_{[k]}>t]dt \geq \int_x^\infty P[X_{[n]}+\ldots+X_{[k]}>t]dt$

for all $x \geq 0$ and $k = 1,2,\ldots,n$.

b) $(EY_{[1]},\ldots,EY_{[n]}) \overset{m}{>} (EX_{[1]},\ldots,EX_{[n]})$.

<u>Proof</u>. b) follows immediately from a). In what follows $\underline{\varepsilon} = (\varepsilon_1,\ldots,\varepsilon_n)$ will denote any vector whose components are zeroes or ones. For $i = 1,\ldots,n$, we define $\phi_i:[0,1]^n \to [0,+\infty)$ by

$$\phi_i(u_1,\ldots,u_n) = \sum_{\underline{\varepsilon},\varepsilon_1+\ldots+\varepsilon_n \geq n-i+1} u_1^{\varepsilon_1}\ldots u_n^{\varepsilon_n}(1-u_1)^{1-\varepsilon_1}\ldots(1-u_n)^{1-\varepsilon_n} .$$

We note that $EX_{[i]} = \int_0^\infty \phi_i(\overline{F}_1(t),\ldots,\overline{F}_n(t))dt$ for $i = 1,\ldots,n$. Now for $k = 1,\ldots,n$ we define

$$\Phi_k(u_1,\ldots,u_n) = \sum_{i=k}^n \phi_i(u_1,\ldots,u_n)$$

$$= \sum_{i=k}^n \sum_{\underline{\varepsilon},\varepsilon_1+\ldots+\varepsilon_n \geq n-i+1} u_1^{\varepsilon_1}\ldots u_n^{\varepsilon_n}(1-u_1)^{1-\varepsilon_1}\ldots(1-u_n)^{1-\varepsilon_n}$$

$$= \sum_{j=1}^n \min(j,n-k+1) \sum_{\underline{\varepsilon},\varepsilon_1+\ldots+\varepsilon_n=j} u_1^{\varepsilon_1}\ldots u_n^{\varepsilon_n}(1-u_1)^{1-\varepsilon_1}\ldots(1-u_n)^{1-\varepsilon_n} .$$

Since $\int_x^\infty P[X_{[n]}+\ldots+X_{[k]} > t]dt = \int_x^\infty \Phi_k(\overline{F}_1(t),\ldots,\overline{F}_n(t))dt$,

it suffices by Theorem 2.1 b to show that each $\Phi_k$ satisfies (2.3) and is concave increasing in each variable separately.

$$\text{Now } \frac{\partial}{\partial u_1}\Phi_k(u_1,\ldots,u_n) = \sum_{j=0}^{n-k} \sum_{\underline{\varepsilon}_1,\varepsilon_2+\ldots+\varepsilon_n=j} u_2^{\varepsilon_2}\ldots u_n^{\varepsilon_n}(1-u_2)^{1-\varepsilon_2}\ldots(1-u_n)^{1-\varepsilon_n}$$

where $\underline{\varepsilon}_1$ represents an n-1 component vector of zeroes and ones.

As $\Phi_k(u_1,\ldots,u_n)$ is symmetric in $u_1,\ldots,u_n$, it follows that $\Phi_k$ is an increasing function linear (and hence concave) in each variable separately. For a continuously twice differentiable function $\Phi$ on $[0,1]^n$, it is easy to verify that the following conditions are equivalent (see Lorentz [7]):

(3.1) $$\Phi(u_i+h, u_j+k) - \Phi(u_i+h, u_j) - \Phi(u_i, u_j+k) + \Phi(u_i, u_j) \geq 0$$

for all $i\neq j$, $0 \leq u_i \leq u_i+h \leq 1$, $0 \leq u_j \leq u_j+k \leq 1$.

(3.2) $$\Phi(u_i+h, u_j+h) - \Phi(u_i+h, u_j) - \Phi(u_i, u_j+h) + \Phi(u_i, u_j) \geq 0$$

for all $i\neq j$, $0 \leq u_i \leq u_i+h \leq 1$, $0 \leq u_j \leq u_j+h \leq 1$.

(3.3) $$\frac{\partial}{\partial u_i}\frac{\partial}{\partial u_j}\Phi(u_1,\ldots,u_n) \geq 0$$

for all $i\neq j$.

Therefore, due to the symmetry of $\Phi_k$ and the above equivalence, it suffices to note that

$$\frac{\partial}{\partial u_1}\frac{\partial}{\partial u_2}\Phi_k(u_1,\ldots,u_n) = -\sum_{\underline{\varepsilon}_{12},\varepsilon_3+\cdots+\varepsilon_n=n-k} u_3^{\varepsilon_3}\cdots u_n^{\varepsilon_n}(1-u_3)^{1-\varepsilon_3}\cdots(1-u_n)^{1-\varepsilon_n}$$
$$\leq 0$$

(where $\underline{\varepsilon}_{12}$ represents an n-2 component vector of zeroes and ones).

<u>Remark 3.2</u>. Let $(X_1,\ldots,X_n)$ and $(Y_1,\ldots,Y_n)$ be random samples of size n from populations with life distribution functions F and G respectively. Barlow and Proschan [1] show that if $G \underset{*}{>} F$ where G and F have common mean, then

$$(EY_{[1]},\ldots,EY_{[n]}) \overset{m}{>} (EX_{[1]},\ldots,EX_{[n]}).$$

Shaked [13] proves the same result under the more *general assumption that* $G \overset{m}{>} F$. His proof uses the characterization of Corollary 2.3 together with the fact that

$$\Psi_k(t_1,\ldots,t_n) = t_{[n]}+\ldots+ t_{[k]}$$

is (separately) convex for each k. It follows that

$$\begin{aligned} EY_{[n]}+\ldots+ EY_{[k]} &= \int_0^\infty \Psi_k(t_1,\ldots,t_n)dG(t_1)\ldots dG(t_n) \\ &\geq \int_0^\infty \Psi_k(t_1,\ldots,t_n)dF(t_1)\ldots dF(t_n) \\ &= EX_{[n]}+\ldots+ EX_{[k]}. \end{aligned}$$

<u>Remark 3.3</u> Suppose that for each $a \in A$, $F^{(a)}$ is distribution function on R, and that $\gamma$ is a probability measure defined on a $\sigma$-field of subsets of A. One may define the n-variate distribution function(assuming appropriate measurability conditions on $F^{(a)}$)

$$F(x_1,\ldots,x_n) = \int_A F^{(a)}(x_1) \ \ldots \ F^{(a)}(x_n)d\gamma(a).$$

If random variables $X_1,\ldots,X_n$ have such a joint distribution function, they are said to be 'positively dependent by mixture'. Given $X_1,\ldots,X_n$ positively dependent by mixture, let $Y_1,\ldots,Y_n$ be independent random variables where $Y_i$ is distributed as $X_i$ for $i = 1,\ldots,n$. Shaked [12] (See also Marshall and Olkin [9] and Proschan [10]) has shown that in this case

$$(EY_{[1]},\ldots,EY_{[n]}) \overset{m}{>} (EX_{[1]},\ldots,EX_{[n]}).$$

<u>Remark 3.4</u> Theorem 3.1 shows that if $G_i \overset{m}{>} F_i$ for all $i=1,\ldots,n$, then for any k $\sum_{i=k}^{n} Y_{[i]}$ is "more variable" than $\sum_{i=k}^{n} X_{[i]}$ (*in the terminology of* Ross [11]) or that $\sum_{i=k}^{n} Y_{[i]}$ is "larger in mean residual life" than $\sum_{i=k}^{n} X_{[i]}$

(in the terminology of Stoyan [14]). Since $\Psi_k(t_1,\ldots,t_n) = t_{[n]}+\ldots+ t_{[k]}$ is convex, this also follows by using the result that if $X_1,\ldots X_n, Y_1, \ldots, Y_n$ are independent and $Y_i$ is "more variable" than $X_i$ for $i=1,\ldots,n$, then $\Psi_k(Y_1,\ldots,Y_n)$ is "more variable" than $\Psi_k(X_1,\ldots,X_n)$ (see Ross [11]).

Remark 3.5 If $X_1,\ldots,X_n$ are independent HNBUE random variables, then Theorem 3.1 b could be useful in constructing bounds on the expected order statistics $EX_{[1]},\ldots,EX_{[n]}$.

Example 3.6 Let us consider the following problem of general interest. n components are to be purchased in order to form a coherent system (for example a k out of n system), and all of the components are to be purchased from either company A or company B. Let us suppose that each company makes the claim that components of type i have mean life $\mu_i$ $(i=1,\ldots,n)$, but that company B is known to be 'more variable' than company A in the production of any type of component. If we wish to maximize the mean life of the system, from which company should we buy?

Let $X_1,\ldots,X_n$ and $Y_1,\ldots,Y_n$ be random variables representing the lifetimes of the components from A and B respectively. If we can assume that the components function independently within the system and that $Y_i$ is more variable than $X_i$ in the sense that $G_i \overset{m}{>} F_i$ (where $X_i \sim F_i$ and $Y_i \sim G_i$) for all $i=1,\ldots,n$, then we know that

$$(EY_{[1]},\ldots, EY_{[n]}) \overset{m}{>} (EX_{[1]},\ldots, EX_{[n]}).$$

In particular $EY_{[1]}- EX_{[1]} \le 0$ and $EY_{[n]} - EX_{[n]} \ge 0$. Therefore if our system is a series system we would buy from A, while if it is parallel we would buy from B. This result was observed by Marshall and Proschan [8].

For a more general k out of n system, we would be interested in the expected order statistics $EX_{[n-k+1]}$ and $EY_{[n-k+1]}$ in order to compare companies A and B. Although

$$(EY_{[1]},\ldots,EY_{[n]}) \overset{m}{>} (EX_{[1]},\ldots,EX_{[n]}),$$

$EY_{[i]} - EX_{[i]}$ may theoretically at least undergo many sign changes as $i:1 \to n$ even in the case when $F_i = F$ and $G_i = G$ for all $i=1,\ldots,n$. However under the assumption that $G \overset{m}{>} F$ where G and F are continuous, G is strictly increasing on its interval support and $G(0)=F(0)=0$, one may show that the number of sign changes in $EY_{[i]} - EX_{[i]}$ is no greater than the number of sign changes in $\overline{G}(x)-\overline{F}(x)$ as $x:0 \to \infty$. Since $\binom{n-1}{i-1} F^{i-1}(t)\,\overline{F}^{n-i}(t)$ is totally positive of order $\infty$ in i and t, this follows using the variation diminishing property of totally positive functions and the identity

$$EY_{[i]} - EX_{[i]} = \int_0^\infty n(G^{-1}F(t) - t)\binom{n-1}{i-1} F^{i-1}(t)\overline{F}^{n-i}(t)dt$$

(see Barlow and Proschan [1]). In particular if $\overline{F}$ crosses $\overline{G}$ once then there exists a constant C (depending on n, F and G) such that

$$EY_{[i]} - EX_{[i]} \le 0 \qquad \text{for } i < C$$

and

$$EY_{[i]} - EX_{[i]} \ge 0 \qquad \text{for } i > C.$$

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